Announcements

1) Math major pizza/t-shirt party, 12-1:30 Monday, 2047 CB (Math Library)

2) Exam Thursday covering Chapter 12, Sections 13.1, 13.2 and arclength in 13.3. Review and Mathematica Lab Tuesday

Integration of Vector Functions

Just like limits and derivatives, to integrate a vector-valued function, integrate each coordinate: if

$$f(t) = \langle x(t), y(t), z(t) \rangle,$$ then

$$\boxed{\int f(t)\,dt = \left\langle \int x(t)\,dt, \int y(t)\,dt, \int z(t)\,dt \right\rangle}$$

provided all integrals exist!

Annoying point: you now have to add a different arbitrary constant in every coordinate!

Example 1: Find the integral of

$$f(t) = \langle \tan(t), t^2\cos(t), \sin^2(t) \rangle$$

$$x(t) = \tan(t)$$

$$\int \tan(t)\,dt = \int \frac{\sin(t)}{\cos(t)}\,dt$$

Let $u = \cos(t)$, $du = -\sin(t)dt$.

Then $\displaystyle\int \frac{\sin(t)}{\cos(t)}\,dt = -\int \frac{1}{u}\,du$

$$= -\ln(u) + C_1$$

Substituting u back in,

we get

$$\boxed{-\ln(\cos(t)) + C_1}$$

Now $y(t) = t^2\cos(t)$,

$\int t^2 \cos(t)\,dt$ uses

integration by parts!

Tabular Trick: choose

$u = t^2$, $dv = \cos(t)$.

Write a table

$u = t^2$	$dv = \cos(t)$

Differentiate $u = t^2$ until you get to zero:

$u = t^2$	$dv = \cos(t)$
$2t$	
2	
0	

Then integrate $dv = \cos(t)$ as many times as you differentiated $u = t^2$

$u = t^2$	$dv = \cos(t)$
$2t$	$\sin(t)$
2	$-\cos(t)$
0	$-\sin(t)$

Now multiply diagonally, alternating signs and starting with "+":

$u = t^2$		$dv = \cos(t)$
$2t$	$+$	$\sin(t)$
2	$-$	$-\cos(t)$
0	$+$	$-\sin(t)$

The individual products are $t^2\sin(t)$, $2t\cos(t)$, and $-2\sin(t)$.

Finally, add all the products together to obtain the antiderivative!

$$\int t^2 \cos(t)\,dt$$

$$= \boxed{t^2 \sin(t) + 2t\cos(t) - 2\sin(t) + C_2}$$

Finally, $z(t) = \sin^2(t)$.

Use the trig identity

$$\sin^2(t) = \frac{1-\cos(2t)}{2}.$$

Then

$$\int \sin^2(t)\,dt = \int \left(\frac{1-\cos(2t)}{2} \right) dt$$

$$= \boxed{\frac{t}{2} - \frac{\sin(2t)}{4} + C_3}$$

The final answer is

$$\int f(t) =$$

$$\boxed{\left\langle -\ln(\cos(t)) + C_1,\ t^2\sin(t) + 2t\cos(t) - 2\sin(t) + C_2,\ \frac{t}{2} - \sin(2t) + C_3 \right\rangle}$$

For definite integration, just add bounds in every integral:

if $f(t) = \langle x(t), y(t), z(t) \rangle$,

then

$$\int_a^b f(t)\,dt =$$

$$\boxed{\left\langle \int_a^b x(t)\,dt, \int_a^b y(t)\,dt, \int_a^b z(t)\,dt \right\rangle}$$

provided all these integrals exist!

Arc length and Curvature

(Section 13.3)

Recall Calc II Formula

If $f(t) = \langle x(t), y(t) \rangle$,

then the arclength of the graph of f from $t=a$ to $t=b$ is given by

$$\boxed{L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt}$$

For this formula to hold, we require that $x'(t), y'(t)$ exist and are continuous on $[a,b]$.

Q: Why is this formula valid?

Justification (kind of)

The idea is to pick some points on the graph and draw line segments in between them. The length of each segment is given by the distance formula applied to the endpoints of the segment.

Thats where the squares and square root in the integral come from.

You add up all the segment lengths to approximate the length of the curve, then you do the usual calculus trick of letting the lengths go to zero as the number of segments go to infinity. This produces the integral and derivatives in the formula.

Arclength Formula in 3-D

The only thing that changes is the appearance of a third coordinate!

If $f(t) = \langle x(t), y(t), z(t) \rangle$, then then the arclength of the graph of f from $t=a$ to $t=b$ is given by

$$\boxed{L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt}$$

Again, we require existence and continuity of x', y', and z' on $[a,b]$.

Example 2: Find the arclength

of

$$g(t) = \langle 2t^2, -5t^2, 3t^3 \rangle$$

from $t=1$ to $t=2$.

$x(t) = 2t^2, \quad x'(t) = 4t$

$y(t) = -5t^2, \quad y'(t) = -10t$

$z(t) = 3t^2, \quad z'(t) = 9t^2.$

Then

$$L = \int_1^2 \sqrt{(4t)^2 + (-10t)^2 + (9t^2)^2}\, dt$$

$$= \int_1^2 \sqrt{16t^2 + 100t^2 + 81t^4}\, dt$$

$$= \int_1^2 \sqrt{t^2(116 + 81t^2)}\, dt$$

$$= \int_1^2 t\sqrt{116 + 81t^2}\, dt$$

Let $u = 116 + 81t^2$, $du = 162t\,dt$

$u(1) = 197,$

$u(2) = 440,$ so the integral is

$$L = \frac{1}{162} \int_{197}^{440} \sqrt{u}\, du$$

$$= \frac{1}{162} \left. \frac{2u^{3/2}}{3} \right|_{197}^{440}$$

$$= \boxed{\frac{1}{241}\left(440^{3/2} - 197^{3/2}\right)}$$

Example 3:

Find the arclength of

$$f(t) = \langle e^{3t}\sin(3t),\ e^{3t}\cos(3t),\ e^{3t} \rangle$$

from $t = 0$ to $t = \ln(2)$.

The derivatives here are much more annoying, but the integral is easier!

1) $x(t) = e^{3t}\sin(3t)$

$x'(t) = 3e^{3t}\sin(3t) + 3e^{3t}\cos(3t)$

2) $y(t) = e^{3t}\cos(3t)$,

$y'(t) = 3e^{3t}\cos(3t) - 3e^{3t}\sin(3t)$

3) $z(t) = e^{3t}$,

$z'(t) = 3e^{3t}$

Now square and add!

$$(x'(t))^2 = 9e^{6t}\cos^2(3t) + 9e^{6t}\sin^2(3t) + \cancel{18e^{6t}\cos(3t)\sin(3t)}$$

$$+ \qquad\qquad\qquad\qquad +$$

$$(y'(t))^2 = 9e^{6t}\cos^2(3t) + 9e^{6t}\sin^2(3t) - \cancel{18e^{6t}\cos(3t)\sin(3t)}$$

$$+ \qquad\qquad\qquad\qquad +$$

$$(z'(t))^2 = \qquad 9e^{6t}$$

$$= 18e^{6t}\left(\underbrace{\cos^2(3t) + \sin^2(3t)}_{=1}\right) + 9e^{6t}$$

$$= 27e^{6t}$$

Then

$$L = \int_0^{\ln(2)} \sqrt{27e^{6t}}\,dt$$

$$= 3\sqrt{3} \int_0^{\ln(2)} e^{3t}\,dt$$

$$= 3\sqrt{3}\left(\frac{e^{3t}}{6} \Big|_0^{\ln(2)} \right)$$

$$= \frac{\sqrt{3}}{2}(8 - 1)$$

$$= \boxed{\frac{7\sqrt{3}}{2}}$$